

## Non-Abelian coset string backgrounds from asymptotic and initial data

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**ABSTRACT:** We describe hierarchies of exact string backgrounds obtained as non-Abelian cosets of orthogonal groups and having a space-time realization in terms of gauged WZW models. For each member in these hierarchies, the target-space backgrounds are identified with the “boundary” backgrounds of the next member. We explicitly demonstrate that this property holds to all orders in  $\alpha'$  for the three- and four-dimensional cosets, while the general structure of the backgrounds at hand suggests that the property holds in any dimension. The affiliation of the “boundary” theory to the “bulk” theory exhibits marginal operators, generically build on non-Abelian parafermion bilinears, dressed with a dilaton vertex operator. The dilaton is supported by the extra radial dimension, whose asymptotic value defines the boundary. Depending on the hierarchy, this boundary can be time-like or space-like with, in the latter case, potential cosmological applications.

**KEYWORDS:** Conformal Field Models in String Theory, Sigma Models.

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## 1. Introduction

In the string literature there has been a class of exact models describing consistent string propagation based on  $G/H$  conformal field theory (CFT) coset models [1]. These have a space-time realization in terms of gauged Wess-Zumino-Witten (WZW) models [2]. In the gauged WZW models the action of the subgroup is generically vectorial and hence fundamentally different than in geometric cosets. Therefore, in order to distinguish them from geometric cosets we will call them henceforth *conformal cosets*. When a certain procedure is followed, a metric, a dilaton and an antisymmetric tensor field arise and the conditions for conformal invariance are satisfied.<sup>1</sup> In the present work we revisit certain conformal coset models based on orthogonal groups.

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<sup>1</sup>This procedure was first carried out explicitly for the two-dimensional solution corresponding to the coset  $SU(2)/U(1)$  completely at the classical level in [3]. The inclusion of the dilaton, absolutely necessary to satisfy conformal invariance, was done for the coset  $SL(2, \mathbb{R})/U(1)$  in [4], where the important physical interpretation of the target space as a two-dimensional black hole was also given. The same solutions were also found in [5, 6] utilizing the beta-functions for conformal invariance.

In particular, we will consider the Euclidean-signature conformal cosets

$$CH_{d,k} = \frac{SO(d, 1)_{-k}}{SO(d)_{-k}}, \quad d = 2, 3, \dots \quad (1.1)$$

and

$$CS_{d,k} = \frac{SO(d + 1)_k}{SO(d)_k}, \quad d = 2, 3, \dots \quad (1.2)$$

In addition, we will consider the corresponding Minkowskian-signature ones

$$CAdS_{d,k} = \frac{SO(d - 1, 2)_{-k}}{SO(d - 1, 1)_{-k}}, \quad d = 2, 3, \dots \quad (1.3)$$

and

$$CdS_{d,k} = \frac{SO(d, 1)_k}{SO(d - 1, 1)_k}, \quad d = 2, 3, \dots \quad (1.4)$$

The indices  $k$  and  $-k$  indicate the level of the corresponding current algebras and will some times be omitted in order to simplify the notation, if there is no risk of confusion. We have given them names reminiscent of the corresponding geometric cosets (spheres, anti-de or de Sitter and hyperbolic planes), but we should keep in mind that *they are different in various respects*.<sup>2</sup> In order to avoid a confusion between geometric and conformal cosets, we have added the letter “C” in the name of the latter, besides the level indices  $k$  or  $-k$ .

The levels in (1.1) and (1.3) are  $-k$  for ensuring that only a single time coordinate appears. The class of models in (1.3) was suggested and analyzed in [11] from an algebraic view point using non-compact current algebras. For the lower-dimensional cases the explicit forms of the corresponding string theory backgrounds have been explicitly worked out to lowest order in  $\alpha' \sim 1/k$ , for  $d = 2$  in [4] and for  $d = 3, 4$  in [12–14]. Moreover, all perturbative  $\alpha'$ -corrections can be systematically worked out using a combination of algebraic CFT and space-time techniques developed in full generality in [15]. Explicit results are worked out for the case with  $d = 2$  in [15, 16] and for  $d = 3, 4$  in [15].

In this paper we will mainly focus on the cosets (1.1)–(1.4) aiming at uncovering their possible geometrical and CFT relations. We will perform most of the computational details for the backgrounds corresponding to the Euclidean-signature conformal cosets. The conclusions we will draw for the Minkowskian-signature ones are easily derived along essentially the same lines. We claim that the spatial infinity of the space for the non-compact coset  $CH_d$  is the space for the compact coset  $CS_{d-1}$  times the linear dilaton  $\mathbb{R}_Q$ , where  $Q$  is an appropriate background charge. This will be explained in detail in

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<sup>2</sup>In geometric cosets, the action of the subgroup is one-sided. Considered as target spaces of sigma-models, geometric cosets lead in general to non-vanishing  $\beta$ -functions and cannot therefore describe consistent string propagation. The exceptions include the three-dimensional anti-de Sitter and the three-sphere, because these are also group manifolds, and cases where, dividing by a subgroup of the Cartan torus, conformal invariance is restored by switching on  $U(1)$  background gauge fields [7–9]. Combinations of maximally symmetric geometric cosets (anti-de Sitter spaces or spheres), as those emerging in near-horizon geometries of brane distributions, can appear as supergravity solutions. We also note the work in [10] where adding torsion to some six-dimensional non-maximally symmetric geometric cosets, made possible to retain conformal invariance.

section 2, where we will present the general structure of the models and of our method and in addition for the benefit of the reader in the subsequent discussion, the general expectations concerning all possible relations involving the conformal cosets (1.1)–(1.4).

In sections 3 and 4 we will further support the above claim by explicitly proving our general assertion for the lowest dimensional  $d = 3$  and  $d = 4$  cosets, respectively, which are already non-trivial, especially the four-dimensional one. The computation will involve the detailed comparison of the background fields at the semiclassical as well as exact in  $\alpha'$  levels. In section 5 we will go further and argue that the full coset  $CH_d$  can be thought of as arising via a marginal perturbation involving a bilinear in the chiral and antichiral parafermions of the compact-coset theory  $CS_{d-1}$  dressed appropriately with a vertex operator in the linear dilaton theory  $\mathbb{R}_Q$ . The anomalous dimension of the parafermions is precisely cancelled out by that of the vertex operator, so that the perturbing operator has exactly dimension  $(1, 1)$ . This is a remarkable observation per se since, to our knowledge, there was only one instance so far where parafermions were involved in marginal perturbations [17].

## 2. General structure of the background fields

The large- $k$  regime background fields for the conformal coset models (1.1)–(1.4) follows from the corresponding gauged WZW action after a unitary gauge (see, for instance, [3, 4] for the simplest cases, namely the two-dimensional coset models) that fixes a number of parameters equal to the dimension of the subgroup  $H$  is chosen and the corresponding gauge fields are integrated out. In general, there is only a metric and a dilaton field, whereas the antisymmetric tensor is zero, irrespective of the dimension  $d$ . By construction the one-loop  $\beta$ -function equations are satisfied. Due to the complexity of the procedure, explicit results are known for the semiclassical large- $k$  regime as well as exactly in  $\alpha' \sim 1/k$ , only for the lower-dimensional cases  $d = 2, 3$  and 4. General results for  $d \geq 5$  are also available in [18].

### 2.1 Large- $k$ semiclassical regime

From the unifying treatment of [14, 15] the structure of the metric of the general coset models in (1.1)–(1.4) in the semiclassical large- $k$ -regime is

$$ds_d^2 = 2k \left[ \frac{db^2}{4(b^2 - 1)} + g_{ij}(b, \mathbf{x}) dx^i dx^j \right], \quad i = 1, 2, \dots, d - 1, \quad (2.1)$$

whereas for the dilaton, the corresponding expression is<sup>3</sup>

$$e^{-2\Phi} = e^{-2\Phi_0} (b + 1)(b - 1)^{d-2} F(\mathbf{x}). \quad (2.2)$$

For the case of the  $CH_{d,k}$  coset in which we work out all details for  $d = 3$  and  $d = 4$  in sections 3 and 4 below, the variable  $b$  is non-compact and takes values in the open intervals with  $|b| > 1$ . The set of variables  $x^i$  can be chosen such that they are all compact

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<sup>3</sup>For the lower dimensional cases  $d = 2, 3$  and 4 this can be seen from the explicit forms of the corresponding metrics and dilaton in [14, 15]. Nevertheless it also holds for  $d \geq 5$  [18].

and take values in finite intervals. The metric components  $g_{ij}(b, \mathbf{x})$  and the function  $F(\mathbf{x})$  have specific forms depending on the particular model. In the limit  $b \rightarrow \infty$  this variable decouples and corresponds to a free boson. In particular, letting

$$b = e^{2x}, \tag{2.3}$$

in the limit  $x \rightarrow \infty$  the metric takes the form<sup>4</sup>

$$ds_d^2 = 2k (dx^2 + \hat{g}_{ij}(\mathbf{x}) dx^i dx^j), \tag{2.4}$$

where the relation to the metric components  $g_{i,j}(b, \mathbf{x})$  in (2.1) is

$$\hat{g}_{ij}(\mathbf{x}) = g_{ij}(\infty, \mathbf{x}). \tag{2.5}$$

The case with  $b \rightarrow -\infty$  is equivalent to that with  $b \rightarrow \infty$ , since we may instead of (2.3) define  $b = -e^{2x}$ , then let  $x \rightarrow \infty$  and note that in our models it turns out that  $g_{ij}(-\infty, \mathbf{x}) = g_{ij}(\infty, \mathbf{x})$ . Similarly for the dilaton

$$e^{-2\Phi} = e^{-2\Phi_0} e^{2(d-1)x} F(\mathbf{x}), \quad \text{as } x \rightarrow \infty. \tag{2.6}$$

Since in this limit for generic values of  $\mathbf{x}$  we have that  $e^\Phi \rightarrow 0$ , the large- $b$  limit is a weak-string coupling limit.

To summarize, the asymptotic region of the  $CH_{d,k}$  coset exhibit, in the large- $k$  regime, the  $(d-1)$ -dimensional metric and the corresponding dilaton

$$ds_{(d-1)}^2 = 2k \hat{g}_{ij}(\mathbf{x}) dx^i dx^j, \quad e^{-2\Phi_{(d-1)}} = F(\mathbf{x}). \tag{2.7}$$

Our claim is that this background corresponds to the semiclassical large- $k$  regime background for the compact coset model  $CS_{d-1,k}$ . We will explicitly demonstrate this in the lowest-dimensional cases with  $d = 3, 4$  which already are highly non-trivial, especially the four-dimensional one. This demonstration is not as straightforward as it might seem up to now since it will involve finding the necessary, quite complicated, coordinate transformations that will make the metrics look identical. The Liouville field contributes asymptotically linearly only to the dilaton term as  $-(d-1)x$ .

## 2.2 All- $k$ exactness

In [15] a method to algebraically deduce all  $\alpha' \sim 1/k$  perturbative corrections to the low-energy metric and dilaton was found for any conformal coset. Using this as well as the explicit results for the low-dimensional cases of the general coset models in (1.1)–(1.4) we deduce that the form of the exact metric is

$$ds^2 = 2(k-d+1) \left[ \frac{db^2}{4(b^2-1)} + g_{ij}(b, \mathbf{x}; k) dx^i dx^j \right], \quad i = 1, 2, \dots, d-1, \tag{2.8}$$

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<sup>4</sup>An alternative equivalent way of viewing this limit is to first rescale the variable  $b$  as  $b \rightarrow b/\lambda$  and then take the limit  $\lambda \rightarrow 0$ . In this way the background (2.1) and (2.2) can be thought of as the integrated perturbation of the leading order correction to the  $\lambda \rightarrow 0$  result.

where we have indicated that the  $(d - 1)$ -dimensional metric  $g_{ij}$  depends explicitly on the level  $k$ . Similarly, for the dilaton we deduce that the exact expression is of the form

$$e^{-2\Phi} = e^{-2\Phi_0} \frac{1}{\sqrt{\beta(b, \mathbf{x}; k)}} (b + 1)(b - 1)^{d-2} F(\mathbf{x}), \quad (2.9)$$

where the function  $\beta$  above is such that  $\beta(b, \mathbf{x}; \infty) = 1$ . Using that and the relation to the semiclassical  $(d - 1)$ -dimensional metric appearing in (2.1)  $g_{ij}(b, \mathbf{x}; \infty) = g_{ij}(b, \mathbf{x})$ , we see that the exact background fields (2.8) and (2.9) go smoothly over to the semiclassical counterparts in (2.1) and (2.2). We also note that since the measure given by the combination

$$e^{-2\Phi} \sqrt{\det(G_{\mu\nu})}, \quad (2.10)$$

is  $k$ -independent [15] we have the relation

$$\det(g_{ij}(b, \mathbf{x}; k)) = \beta(b, \mathbf{x}; k) \det(g_{ij}(b, \mathbf{x})). \quad (2.11)$$

We claim that, in the limit  $b \rightarrow \infty$ , the exact matching involves also a shift of the level  $k$  as  $k \rightarrow k + 2d - 4$ . These will be shown by comparing the exact  $\alpha'$ -corrected geometries for the three- and four-dimensional cases in sections 3 and 4, as well as when we examine the nature of the marginal perturbation, involving dressed parafermions in section 5, in any dimension. For the time being let's assume that and recall that the central charges corresponding to the models (1.1) and (1.2) are

$$\begin{aligned} c_{CH}(d, k) &= \frac{d(d+1)k}{2(k-d+1)} - \frac{d(d-1)k}{2(k-d+2)}, \\ c_{CS}(d, k) &= \frac{d(d+1)k}{2(k+d-1)} - \frac{d(d-1)k}{2(k+d-2)}. \end{aligned} \quad (2.12)$$

Then the difference

$$c_{CH}(d, k + 2d - 4) - c_{CS}(d - 1, k) = 1 + \frac{3(d-1)^2}{k + d - 3}, \quad (2.13)$$

corresponds to the central charge of the extra linear dilaton theory  $\mathbb{R}_Q$ . Identifying this with  $1 + 12Q^2$  we read off the background charge as

$$Q_{d,k} = -\frac{d-1}{2\sqrt{k+d-3}}. \quad (2.14)$$

Returning back to the discussion of the structure of the background metric and dilaton (eqs. (2.8) and (2.9)), in the limit  $b \rightarrow \infty$  and after shifting the level  $k \rightarrow k + 2d - 4$  we obtain

$$ds^2 = 2(k + d - 3) \frac{db^2}{4b^2} + 2(k + d - 2) \hat{g}_{ij}(\mathbf{x}; k) dx^i dx^j, \quad (2.15)$$

where

$$\hat{g}_{ij}(\mathbf{x}; k) = \frac{k + d - 3}{k + d - 2} g_{ij}(\infty, \mathbf{x}; k + 2d - 4). \quad (2.16)$$

Similarly for the dilaton we have

$$e^{-2\Phi} = e^{-2\Phi_0} e^{-2(d-1)x} \frac{F(\mathbf{x})}{\sqrt{\hat{\beta}(\mathbf{x}; k)}}, \quad (2.17)$$

with

$$\hat{\beta}(\mathbf{x}; k) = c_{d,k} \beta(\infty, \mathbf{x}; k + 2d - 4), \quad c_{d,k} = \left( \frac{k + d - 3}{k + d - 2} \right)^{d-1}, \quad (2.18)$$

where the value of the constant  $c_{d,k}$  is dictated by the right limiting behaviour as  $b \rightarrow \infty$  as well as the relation (2.11). The asymptotic region of the exact  $d$ -dimensional background at hand finally reads

$$ds_{(d-1)}^2 = 2(k + d - 2) \hat{g}_{ij}(\mathbf{x}) dx^i dx^j, \quad e^{-2\Phi_{(d-1)}} = \frac{F(\mathbf{x})}{\sqrt{\hat{\beta}(\mathbf{x}; k)}}. \quad (2.19)$$

As we will prove for  $d = 3$  and  $4$ , these  $(d - 1)$ -dimensional metric and dilaton correspond to the exact backgrounds for the compact coset model  $CS_{d-1,k}$ . We expect this to be true for the higher dimensional cases as well.

### 2.3 General expectations

In conclusion, we schematically expect the following relation to hold for any  $d$

$$CH_{d,k+2d-4} \text{ at spatial infinity becomes } CS_{d-1,k} \times \mathbb{R}_{Q_{d,k}}, \quad (2.20)$$

where we have emphasized the shift in the level  $k$ . Besides the case of Euclidean-signature cosets we also note the relation between the Minkowskian-signature cosets

$$CAdS_{d,k+2d-4} \text{ at spatial infinity becomes } CdS_{d-1,k} \times \mathbb{R}_{Q_{d,k}}. \quad (2.21)$$

The above two cases are obtained by considering the spatial asymptotic regions of the higher-dimensional coset spaces and the relation is between spaces of the same signature. By considering a limiting procedure that involves the time variable going to the infinite past (or future) we may obtain the following relation between a Minkowskian-signature and a Euclidean-signature coset

$$CdS_{d,k-2d+4} \text{ at temporal infinity becomes } CH_{d-1,k} \times \mathbb{R}_{\tilde{Q}_{d,k}}, \quad (2.22)$$

where in the latter case the Liouville field is timelike, so that its central charge is  $1 - 12\tilde{Q}^2$ , with

$$\tilde{Q}_{d,k} = -\frac{d-1}{2\sqrt{k-d+3}}. \quad (2.23)$$

### 3. The $d = 3$ example

In this section we explicitly verify our claim (2.20) at the level of the semiclassical and exact background fields for the three-dimensional case.

### 3.1 The large- $k$ regime

For the three-dimensional case the metric and dilaton in the large- $k$  semiclassical limit were found for various patches of the global space in [12]. For our purposes it is convenient to use global coordinates which have the appropriate ranges to cover the entire manifold and according to their range they can describe all different coset cases (1.1)–(1.4) (for  $d = 3$ ). These variables have a group-theoretical origin, relating to invariants of the gauge subgroup and were constructed in [14]. We have for the metric

$$ds^2 = 2k \left( \frac{d\hat{b}^2}{4(\hat{b}^2 - 1)} + \frac{\hat{b} - 1}{\hat{b} + 1} \frac{d\hat{u}^2}{4\hat{u}(\hat{v} - \hat{u} - 2)} - \frac{\hat{b} + 1}{\hat{b} - 1} \frac{d\hat{v}^2}{4\hat{v}(\hat{v} - \hat{u} - 2)} \right) \quad (3.1)$$

and for the dilaton

$$e^{-2\Phi} = e^{-2\Phi_0} |(\hat{b}^2 - 1)(\hat{v} - \hat{u} - 2)|. \quad (3.2)$$

They are indeed of the forms (2.1) and (2.2), where, for later convenience, we have used the notation  $\hat{b}$  instead of just  $b$ .

Depending on the ranges of the real variables  $\hat{u}, \hat{v}$  and  $\hat{b}$ , this background corresponds to either of the cosets (1.1)–(1.4) for  $d = 3$ . Specifically [14]

$$\begin{aligned} \text{CAdS}_3 &: \left\{ |\hat{b}| > 1, \hat{u}\hat{v} > 0 \right\} \quad \text{or} \quad \left\{ |\hat{b}| < 1, \hat{u}\hat{v} < 0 \text{ excluding } 0 < \hat{v} < \hat{u} + 2 < 2 \right\}, \\ \text{CdS}_3 &: \left\{ |\hat{b}| > 1, \hat{u}\hat{v} < 0 \text{ excluding } 0 < \hat{v} < \hat{u} + 2 < 2 \right\} \quad \text{or} \quad \left\{ |\hat{b}| < 1, \hat{u}\hat{v} > 0 \right\}, \\ \text{CH}_3 &: \left\{ |\hat{b}| > 1, 0 < \hat{v} < \hat{u} + 2 < 2 \right\}, \\ \text{CS}_3 &: \left\{ |\hat{b}| < 1, 0 < \hat{v} < \hat{u} + 2 < 2 \right\}. \end{aligned} \quad (3.3)$$

The level  $k$  appearing in (3.1) is assumed to be positive for the cosets  $\text{CAdS}_{d,k}$  and  $\text{CH}_{d,k}$ . For the  $\text{CdS}_{d,k}$  and  $\text{CS}_{d,k}$  we should flip its sign, i.e.  $k \rightarrow -k$  in order to have the correct signature. This is the origin of the negative sign in the levels appearing in (1.1) and (1.3).

For later use note also that the above geometry has the following curvature invariants (in all curvature invariants we compute in this paper we have not included the overall  $2k$  factor in the metric):

$$R = 8 \frac{3 + \hat{b}^2 + \hat{u} - \hat{v} - b(\hat{u} + \hat{v})}{(\hat{b}^2 - 1)(\hat{v} - \hat{u} - 2)} \quad (3.4)$$

and

$$\frac{\det g_{\mu\nu}}{\det R_{\mu\nu}} = \frac{1}{64} (\hat{b}^2 - 1) (\hat{v} - \hat{u} - 2). \quad (3.5)$$

The third invariant  $R_{\mu\nu} R^{\mu\nu} = R^2/2$  is not independent in this case. Introduce finally the combination

$$K = 8R \frac{\det g_{\mu\nu}}{\det R_{\mu\nu}} = 3 + \hat{b}^2 + \hat{u} - \hat{v} - \hat{b}(\hat{u} + \hat{v})^2, \quad (3.6)$$



which is also invariant, but not independent. A *higher-order* in derivatives invariant, independent of the three canonical curvature invariants of three-dimensional manifolds is<sup>5</sup>

$$L = \frac{1}{3} \left( \frac{\square}{4} - 1 \right) K = \hat{b}^2 - 1 . \tag{3.7}$$

We restrict to the Euclidean-signature non-compact coset  $CH_{3,k}$ , we let the variable  $\hat{b} \rightarrow \infty$  and simultaneously trade it for  $x$  as  $\hat{b} = e^{2x}$ , with  $x \rightarrow \infty$ , as in (2.3). Then the variable  $x$  in the metric (3.1) decouples whereas in the dilaton (3.2) it contributes the linear term  $-2x$ . Hence for generic values of  $\hat{u}$  and  $\hat{v}$  this is weak-coupling limit. In order to interpret the remaining two-dimensional metric let the coordinate change

$$\hat{u} = -2 \sin^2 \theta \cos^2 \phi , \quad \hat{v} = 2 \sin^2 \theta \sin^2 \phi . \tag{3.8}$$

This covers entirely the allowed range of the variables  $\hat{v}$  and  $\hat{u}$  in (3.3) and the two-dimensional part of the metric becomes

$$ds_{(2)}^2 = 2k(d\theta^2 + \tan^2 \theta d\phi^2) . \tag{3.9}$$

In addition, the corresponding part of the dilaton is

$$e^{-2\Phi_{(2)}} = \cos^2 \theta . \tag{3.10}$$

The background (3.9) and (3.10) corresponds to the compact coset  $CS_{2,k}$  (found for the coset  $SL(2, \mathbb{R})/\mathbb{R}$  in [4] from which it is obtained by a simple analytic continuation and called the *bell geometry*), as advertized above.

### 3.2 The exact background

The exact, in the perturbative  $1/k$  expansion, expression for the metric has the form [15]

$$ds^2 = 2(k-2) \left( G_{\hat{b}\hat{b}} d\hat{b}^2 + G_{\hat{v}\hat{v}} d\hat{v}^2 + G_{\hat{u}\hat{u}} d\hat{u}^2 + 2G_{\hat{v}\hat{u}} d\hat{v} d\hat{u} \right) \tag{3.11}$$

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<sup>5</sup>In general a  $d$ -dimensional manifold has a number of scalars constructed out of the metric and its derivatives. At most  $d$  of such scalars are *functionally* independent. The number of *algebraically* independent scalar invariants, not satisfying any polynomial relation is much larger and was computed in [19], more than 100 years ago. It can be proved, though it is intuitively obvious that there is an equivalent statement to say that an invariant depends on the metric, the curvature tensor and its covariant derivatives up to order  $k$ , called the order of the invariant. Then in [19] it was found that the result for the number of zeroth order invariants, that is those built with the metric and the curvature tensor is (see also for this case p. 145 of [20])

$$J_{d,0} = \frac{1}{12} (d-2)(d-1)d(d+3), \quad d \geq 3 .$$

For higher dimensional invariants that necessarily contain the  $(k+2)$ -th derivative of the metric or equivalently the  $k$ -th covariant derivative of the curvature tensor, the result is

$$I_{d,k} = d \frac{k+1}{2} \frac{(d+k+1)!}{(d-2)!(k+3)!}, \quad k \geq 1, \quad d \geq 3 .$$

There are in general differential relations among these invariants and the number of functionally independent ones is less than  $J_{d,0}$  and  $I_{d,k}$  which should be thought of as upper bounds. For  $d=3$  we have  $J_{3,0} = 3$  and the invariants for our case are listed above. We also have that  $I_{3,2} = 27$ , so that the invariant  $L$  in (3.7) should be related to a linear combination of them.

with

$$\begin{aligned}
 G_{\hat{b}\hat{b}} &= \frac{1}{4(\hat{b}^2 - 1)}, \\
 G_{\hat{u}\hat{u}} &= \frac{\beta(\hat{b}, \hat{u}, \hat{v})}{4\hat{u}(\hat{v} - \hat{u} - 2)} \left( \frac{\hat{b} - 1}{\hat{b} + 1} - \frac{1}{k - 1} \frac{\hat{v} - 2}{\hat{v} - \hat{u} - 2} \right), \\
 G_{\hat{v}\hat{v}} &= -\frac{\beta(\hat{b}, \hat{u}, \hat{v})}{4\hat{v}(\hat{v} - \hat{u} - 2)} \left( \frac{\hat{b} + 1}{\hat{b} - 1} + \frac{1}{k - 1} \frac{\hat{u} + 2}{\hat{v} - \hat{u} - 2} \right), \\
 G_{\hat{v}\hat{u}} &= \frac{1}{4(k - 1)} \frac{\beta(\hat{b}, \hat{u}, \hat{v})}{(\hat{v} - \hat{u} - 2)^2},
 \end{aligned} \tag{3.12}$$

where the function  $\beta(\hat{b}, \hat{u}, \hat{v})$  (compared to the general notation we have adopted so far, we omit displaying  $k$ -dependence explicitly) is defined as

$$\beta^{-1}(\hat{b}, \hat{u}, \hat{v}) = 1 + \frac{1}{k - 1} \frac{1}{\hat{v} - \hat{u} - 2} \left( \frac{\hat{b} - 1}{\hat{b} + 1} (\hat{u} + 2) - \frac{\hat{b} + 1}{\hat{b} - 1} (\hat{v} - 2) - \frac{2}{k - 1} \right). \tag{3.13}$$

In addition the exact dilaton is

$$e^{-2\Phi} = e^{-2\Phi_0} \frac{|(\hat{b}^2 - 1)(\hat{v} - \hat{u} - 2)|}{\sqrt{\beta(\hat{b}, \hat{u}, \hat{v})}}. \tag{3.14}$$

Indeed, (3.11) and (3.14) are of the form (2.8) and (2.9). In the limit  $x \rightarrow \infty$  ( $\hat{b} = e^{2x}$ ), the variable  $x$  decouples as before and similarly contributes a term  $-2x$  to the dilaton field. The remaining part of the metric, after the shift  $k \rightarrow k + 2$  and the coordinate change (3.8), becomes

$$ds_{(2)}^2 = 2(k + 1) \left( d\theta^2 + \left(1 - \frac{1}{k} \tan^2 \theta\right)^{-1} \tan^2 \theta d\phi^2 \right), \tag{3.15}$$

whereas the corresponding exact dilaton becomes

$$e^{-2\Phi_{(2)}} = \left(1 - \frac{1}{k} \tan^2 \theta\right)^{1/2} \cos^2 \theta. \tag{3.16}$$

This is nothing but the exact expressions for the metric and dilaton corresponding to the  $CS_{2,k}$  coset [15, 16].

#### 4. The $d = 4$ example

In this section we explicitly verify our claim (2.20) at the level of the semiclassical and exact background fields for the four-dimensional case. This will be a highly non-trivial check as we shall see.

### 4.1 The large- $k$ regime

For the four-dimensional cases the metric and dilaton in the large- $k$  semiclassical limit were found for various patches of the global space in [13]. Again we use here the expression that can cover as before the global space for all different cosets by appropriately restricting the ranges of the corresponding variables [14, 15]. The metric and the dilaton are

$$ds^2 = 2k \left( \frac{db^2}{4(b^2 - 1)} + \frac{b - 1}{b + 1} \frac{du^2}{4(v - u)(u - w)} + \frac{b + 1}{b - 1} \frac{v - w}{4} \left[ \frac{dw^2}{(1 - w^2)(u - w)} - \frac{dv^2}{(v^2 - 1)(v - u)} \right] \right) \quad (4.1)$$

and

$$e^{-2\Phi} = e^{-2\Phi_0} |(b^2 - 1)(b - 1)(v - u)(w - u)|, \quad (4.2)$$

both of them of the form (3.1) and (3.2). As in three dimensions, the signature is determined by the ranges of the four variables  $u, v, w$  and  $b$ . The analog of (3.3) is, as expected, more complicated and we restrict our presentation on that for the coset  $CH_{4,k}$  (For the other cases the interested reader should consult [15]). The correct ranges of variables are

$$-1 < v < u < w < 1, \quad |b| > 1, \quad \text{or} \quad -1 < w < u < v < 1, \quad |b| > 1. \quad (4.3)$$

This follows from the analysis and results of [15] for this particular case, in which the ranges were found utilizing the group theoretical origin of the target-space variables. Since the metric is manifestly invariant under the interchange of the variables  $v$  and  $w$  we may restrict our discussion to the range

$$-1 < v < u < w < 1, \quad |b| > 1, \quad (4.4)$$

with no loss of generality. In the limit  $b = e^{2x} \rightarrow \infty$  the coordinate  $x$  decouples from the metric and it contributes the linear term  $-3x$  to the dilaton. The remaining part of the metric has the form

$$ds_{(3)}^2 = 2k \left[ \frac{du^2}{4(v - u)(u - w)} + \frac{v - w}{4} \left( \frac{dw^2}{(1 - w^2)(u - w)} - \frac{dv^2}{(v^2 - 1)(v - u)} \right) \right]. \quad (4.5)$$

In addition the corresponding part of the dilaton is

$$e^{-2\Phi_{(3)}} = |(v - u)(w - u)|. \quad (4.6)$$

This limiting three-dimensional geometry has the following curvature invariants in one to one correspondence with (3.4)–(3.7):

$$R = 4 \frac{1 - 2vw + (u - v - w)^2}{(u - v)(u - w)}, \quad (4.7)$$

$$\frac{\det g_{\mu\nu}}{\det R_{\mu\nu}} = \frac{(u - v)(u - w)}{32}, \quad (4.8)$$

$$K = 8R \frac{\det g_{\mu\nu}}{\det R_{\mu\nu}} = 1 - 2vw + (u - v - w)^2 \quad (4.9)$$

and

$$L = \frac{1}{3} \left( \frac{\square}{4} + 1 \right) K = - (u - v - w + 1) (u - v - w - 1) , \quad (4.10)$$

where we note a convenient flip of the relative sign between the two terms in the higher-derivative invariant  $L$  as compared with (3.7). A relation of (3.1), (3.2) with (4.5), (4.6) is not apparent. There is however a coordinate transformation that transforms the metric (3.1) to minus the metric (4.5). This is constructed by comparing the corresponding invariants (3.4), (3.5) and (3.7) with (4.7), (4.8) and (4.10). Since we want eventually the three-dimensional metrics to be opposite to each other, we demand that the corresponding  $R$ 's and  $L$ 's are opposite to each other, whereas the  $K$ 's are equal. This provides three algebraic relations between the two sets of coordinates from which we find

$$\begin{aligned} \hat{b} &= u - v - w , \\ \hat{u} &= - \frac{(1+v)(1+w)}{1-u+v+w} , \\ \hat{v} &= \frac{(1-v)(1-w)}{1+u-v-w} . \end{aligned} \quad (4.11)$$

From this and (4.4) we may easily show that  $\hat{b}, \hat{u}$  and  $\hat{v}$  have the correct ranges as given by the last of (3.3). The inverse transformation is

$$\begin{aligned} u &= \hat{b} + \frac{\hat{b}}{2} (\hat{u} - \hat{v}) - \frac{1}{2} (\hat{u} + \hat{v}) , \\ v + w &= \frac{\hat{b}}{2} (\hat{u} - \hat{v}) - \frac{1}{2} (\hat{u} + \hat{v}) , \\ vu &= \frac{\hat{b}}{2} (\hat{u} + \hat{v}) - \frac{1}{2} (\hat{u} - \hat{v}) - 1 . \end{aligned} \quad (4.12)$$

Then the metric (3.1) indeed transforms into minus (4.5) and accordingly the dilaton. It is worth stressing that the transformation above would have been hard to find without the method of comparing invariants.

## 4.2 The exact background

We will now verify to all orders in  $1/k$  that the asymptotic region of the four-dimensional coset under consideration indeed matches the three-dimensional coset plus a free boson. The exact metric and dilaton are [15]

$$\begin{aligned} ds^2 &= 2(k-3) \left( G_{bb} db^2 + G_{uu} du^2 + G_{vv} dv^2 + G_{ww} dw^2 \right. \\ &\quad \left. + 2G_{uv} du dv + 2G_{uw} du dw + 2G_{vw} dv dw \right) , \end{aligned} \quad (4.13)$$

where

$$\begin{aligned}
G_{bb} &= \frac{1}{4(b^2-1)}, \\
G_{uu} &= \frac{\beta(b, u, v, w)}{4(u-w)(v-u)} \left[ \frac{b-1}{b+1} - \frac{1}{k-2} \frac{(v-w)^2}{(v-u)(u-w)} \left( 1 - \frac{1}{k-2} \frac{b+1}{b-1} \right) \right], \\
G_{vv} &= \frac{(v-w)}{4(v^2-1)(v-u)} \frac{\beta(b, u, v, w)}{4(v-u)} \left[ \frac{b+1}{b-1} - \frac{1}{k-2} \frac{1}{(v-u)(u-w)} \right. \\
&\quad \times \left. \left( 1 - u^2 + \left( \frac{b+1}{b-1} \right)^2 (v-u)(v-w) + \frac{1}{k-2} \frac{b+1}{b-1} \frac{(1+v^2)(u+w) - 2v(1+uw)}{v-w} \right) \right], \\
G_{ww} &= \frac{(v-w)}{4(1-w^2)(u-w)} \frac{\beta(b, u, v, w)}{4(v-u)} \left[ \frac{b+1}{b-1} - \frac{1}{k-2} \frac{1}{(v-u)(u-w)} \right. \\
&\quad \times \left. \left( 1 - u^2 + \left( \frac{b+1}{b-1} \right)^2 (u-w)(v-w) - \frac{1}{k-2} \frac{b+1}{b-1} \frac{(1+w^2)(u+v) - 2w(1+uw)}{v-w} \right) \right], \\
G_{uv} &= \frac{\beta(b, u, v, w)}{4(k-2)(v-u)^2} \left( 1 - \frac{1}{k-2} \frac{b+1}{b-1} \frac{v-w}{u-w} \right), \\
G_{uw} &= \frac{\beta(b, u, v, w)}{4(k-2)(u-w)^2} \left( 1 - \frac{1}{k-2} \frac{b+1}{b-1} \frac{v-w}{v-u} \right), \\
G_{vw} &= \frac{1}{(k-2)^2} \frac{b+1}{b-1} \frac{\beta(b, u, v, w)}{4(v-u)(u-w)},
\end{aligned} \tag{4.14}$$

with the function  $\beta(b, u, v, w)$  (again, we omit displaying  $k$ -dependence explicitly) being defined as

$$\begin{aligned}
\beta^{-1}(b, u, v, w) &= 1 + \frac{1}{k-2} \frac{(v-w)^2}{(v-u)(w-u)} \left[ \frac{b+1}{b-1} + \frac{b-1}{b+1} \frac{1-u^2}{(v-w)^2} + \right. \\
&\quad \left. \frac{1}{k-2} \left( \frac{vw + u(v+w) - 3}{(v-w)^2} - \left( \frac{b+1}{b-1} \right)^2 \right) \right] + \frac{2}{(k-2)^3} \frac{b+1}{b-1} \frac{vw-1}{(v-u)(u-w)}.
\end{aligned} \tag{4.15}$$

The dilaton field is

$$e^{-2\Phi} = e^{-2\Phi_0} \frac{|(b^2-1)(b-1)(v-u)(w-u)|}{\sqrt{\beta(b, u, v, w)}}. \tag{4.16}$$

This background in the limit  $b \rightarrow \infty$  gives rise to a three-dimensional metric, which after the shift  $k \rightarrow k+4$ , can be written in the form

$$\begin{aligned}
ds_{(3)}^2 &= 2(k+1)\beta(\infty, u, v, w) \left[ \left( A_1 + \frac{k+1}{(k+2)^2} A_2 \right) du^2 + \left( B_1 + \frac{B_2}{k+2} + \frac{B_3}{(k+2)^2} \right) dv^2 \right. \\
&\quad + \left( C_1 + \frac{C_2}{k+2} + \frac{C_3}{(k+2)^2} \right) dw^2 + 2 \left( \frac{D_2}{k+2} + \frac{D_3}{(k+2)^2} \right) du dv \\
&\quad \left. + 2 \left( \frac{E_2}{k+2} + \frac{E_3}{(k+2)^2} \right) du dw + 2 \frac{Z_3}{(k+2)^2} dv dw \right],
\end{aligned} \tag{4.17}$$

where the various entries are

$$\begin{aligned}
 A_1 &= \frac{1}{4(u-w)(v-u)}, & A_2 &= -\frac{(v-w)^2}{4(v-u)^2(u-w)^2}, \\
 B_1 &= -\frac{v-w}{4(v^2-1)(v-u)}, & B_2 &= \frac{(v-w)[1-u^2+(v-u)(v-w)]}{4(v^2-1)(v-u)^2(u-w)}, \\
 B_3 &= \frac{(1+v^2)(u+w)-2v(1+uw)}{4(v^2-1)(v-u)^2(u-w)}, \\
 C_1 &= -\frac{w-v}{4(w^2-1)(w-u)}, & C_2 &= \frac{(w-v)[1-u^2+(w-u)(w-v)]}{4(w^2-1)(w-u)^2(u-v)}, \\
 C_3 &= \frac{(1+w^2)(u+v)-2w(1+uv)}{4(w^2-1)(w-u)^2(u-v)}, \\
 D_2 &= \frac{1}{4(v-u)^2}, & D_3 &= -\frac{v-w}{4(v-u)^2(u-w)}, \\
 E_2 &= \frac{1}{4(w-u)^2}, & E_3 &= -\frac{w-v}{(w-u)^2(u-v)}, \\
 Z_3 &= \frac{1}{4(v-u)(u-w)}.
 \end{aligned} \tag{4.18}$$

With these definitions the metric (4.17) can be recast as

$$\begin{aligned}
 ds_{(3)}^2 &= 2(k+2)\beta_\infty(u, v, w) \left[ \left( A_1 + \frac{2A_1 + A_2}{k+1} + \frac{A_1}{(k+1)^2} \right) du^2 \right. \\
 &\quad + \left( B_1 + \frac{2B_1 + B_2}{k+1} + \frac{B_1 + B_2 + B_3}{(k+1)^2} \right) dv^2 \\
 &\quad + \left( C_1 + \frac{2C_1 + C_2}{k+1} + \frac{C_1 + C_2 + C_3}{(k+1)^2} \right) dw^2 + 2 \left( \frac{D_2}{k+1} + \frac{D_2 + D_3}{(k+1)^2} \right) du dv \\
 &\quad \left. + 2 \left( \frac{E_2}{k+1} + \frac{E_2 + E_3}{(k+1)^2} \right) du dw + 2 \frac{Z_3}{(k+1)^2} dv dw \right],
 \end{aligned} \tag{4.19}$$

where  $\beta_\infty$  is defined as

$$\beta^{-1}(\infty, u, v, w) = \left( \frac{k+1}{k+2} \right)^3 \beta_\infty^{-1}(u, v, w), \tag{4.20}$$

where the overall constant is precisely  $c_{4,k}$  in (2.18). We can now check that the above expression for the metric can be obtained from the metric (3.11) under the transformation (4.11) and the sign change  $k \rightarrow -k$  (note that precisely  $\beta_\infty(u, v, w) = \beta(\hat{b}, \hat{u}, \hat{v})$ ). It is remarkable that the coordinate transformation (4.11) does not receive  $1/k$ -corrections, a fact that is attributed to the group-theoretical nature of the coordinates we are using.

## 5. Marginal perturbations and parafermions

So far we have explicitly demonstrated that in three and four dimensions, there exists a limiting procedure in the  $CH_d$  coset involving a radial coordinate taken to spatial infinity, in which this coordinate decouples and the remaining lower-dimensional background corresponds to the coset  $CS_{d-1}$ .

We can recast this observation from a slightly different perspective by saying that the conformal sigma models  $CH_d$  and  $CS_{d-1} \times \mathbb{R}_Q$  have target spaces and dilaton backgrounds that coincide in an asymptotic corner with respect to some non-compact radial coordinate. Hence, these two sigma models are both solutions of the full beta-function equations with common asymptotics. The leading correction of the asymptotic expansion of  $CH_d$  delivers, as we will show below, a marginal operator of  $CS_{d-1} \times \mathbb{R}_Q$ . One therefore expects to recover  $CH_d$  by following the deformation of  $CS_{d-1} \times \mathbb{R}_Q$  induced by that operator, i.e. by integrating the full beta-function equations with the  $CS_{d-1} \times \mathbb{R}_Q$  asymptotics and the radial derivative as prescribed by the marginal operator itself.

In the following, we will compute the leading correction and show that from the  $CS_{d-1} \times \mathbb{R}_Q$  viewpoint, it corresponds to a marginal perturbation involving the non-Abelian parafermions of the  $CS_{d-1}$  coset, for  $d \geq 4$  or the Abelian ones of the  $CS_2$ , for  $d = 3$ , dressed with the Liouville field of  $\mathbb{R}_Q$ .

### 5.1 General considerations

We consider again the expansion of the exact metric (2.8) of  $CH_d$ , but now we keep the leading correction in the  $b \rightarrow \infty$  limit. We get an expression of the form

$$ds^2 = 2(k + d - 3) \frac{db^2}{4b^2} + 2(k + d - 2) \hat{g}_{ij}(\mathbf{x}; k) dx^i dx^j + \frac{1}{b} \xi^i S_{ij} \xi^i + \mathcal{O}\left(\frac{1}{b^2}\right), \quad (5.1)$$

The decoupled variable  $b$  corresponds to a free boson. Instead of the change of variable (2.3), in the discussion of this section it helps to absorb the  $k$ -dependence into the transformation as

$$b = e^{2x} = e^{-a_{d,k} \Phi}, \quad a_{d,k} = -\frac{1}{\sqrt{k + d - 3}}, \quad (5.2)$$

so that the boson  $\Phi$  is properly normalized and has a background charge given by (2.14).

Now we turn our attention to the leading correction term and show that it is indeed a marginal perturbation of the  $CS_{d-1} \times \mathbb{R}_Q$ . This correction term contains, via the factor  $1/b$  a vertex operator of the form

$$V_{d,k} = e^{a_{d,k} \Phi}, \quad (5.3)$$

which has conformal dimension

$$\Delta_{d,k} = \bar{\Delta}_{d,k} = -\frac{a_{d,k}^2}{2} + Q_{d,k} a_{d,k} = \frac{d - 2}{2(k + d - 3)}, \quad (5.4)$$

where in the second equality we have substituted the values for  $a_{d,k}$  and  $Q_{d,k}$  using (5.2) and (2.14). The bilinear  $\xi^i S_{ij} \xi^j$ , where  $\xi^i$  are differentials and  $S_{ij}$  a non-constant, in general, matrix multiplying this vertex, corresponds to the classical parafermions bilinear of the coset  $CS_{d-1,k}$  theory, as we next explain, after a short introduction on those aspects of parafermions, both classical and quantum, that are necessary in this paper.

Parafermionic quantum algebras were first introduced and analyzed in [21] and it turns out that they correspond to natural chiral and antichiral objects in the  $SU(2)/U(1)$

conformal coset theory.<sup>6</sup> Their classical counterparts, called classical parafermions, of a general coset  $G/H$  theory were introduced later in [3, 23] as gauge-invariant objects of the theory. They essentially arise from the currents of the WZW theory for the group  $G$  with coset indices dressed with Wilson lines involving the gauge fields that eventually are integrated out in the path integral in the gauged WZW action. Denoting by Latin (Greek) letters subgroup  $H$  (coset  $G/H$ ) indices, we have a set of holomorphic parafermions  $\Psi^\alpha(\sigma^+)$  and a set of antiholomorphic ones  $\bar{\Psi}^\alpha(\sigma^-)$ . The non-trivial braiding properties are reflected in the classical Poisson bracket algebra they obey [3]

$$\begin{aligned} \{\Psi_\alpha(x), \Psi_\beta(y)\} = & -\frac{k}{\pi} \delta_{\alpha\beta} \delta'(x-y) - f_{\alpha\beta\gamma} \Psi_\gamma(y) \delta(x-y) \\ & - \frac{\pi}{2k} f_{c\alpha\gamma} f_{c\beta\delta} \epsilon(x-y) \Psi_\gamma(x) \Psi_\delta(y), \end{aligned} \tag{5.5}$$

where the antisymmetric step function  $\epsilon(x-y)$  equals  $+1$  ( $-1$ ) if  $x > y$  ( $x < y$ ). The last term in (5.5) is responsible for their non-trivial monodromy properties and unusual statistics. In the above,  $x$  and  $y$  denote world-sheet light-cone variables such as  $\sigma^+$  and  $\sigma'^+$ , so that the Poisson brackets in (5.5) are evaluated at equal light-cone time  $\sigma^-$ . In addition, conformal transformations are generated by  $T_{++} = -\frac{\pi}{2k} \Psi_\alpha \Psi_\alpha$ . A similar algebra to (5.5) and statements are valid for the antiholomorphic parafermions  $\bar{\Psi}^\alpha$  as well.

The classical parafermions have dimension one, but quantum mechanically they receive  $1/k$ -corrections. In [24] the above Poisson algebra was promoted at the level of an OPE conformal algebra and various consistency conditions were checked extensively. In particular, it was found that the anomalous dimension of the parafermions is completely dictated by the structure of the braiding last term in the Poisson algebra (5.5). For the general case the dimension matrix is

$$\Delta_{\alpha\beta} = \bar{\Delta}_{\alpha\beta} = \delta_{\alpha\beta} - \frac{f_{c\gamma\alpha} f_{c\gamma\beta}}{k + g_H}, \tag{5.6}$$

where the dual Coxeter number  $g_H$  for the subgroup is  $f_{acd} f_{bcd} = g_H \delta_{ab}$ . There is an alternative way of seeing that this is the conformal dimension of the parafermions. Recall, that they are essentially the generators  $J_\alpha$  of the currents with coset indices. Then with respect to the energy-momentum tensor of the coset theory

$$T_{G/H} = T_G - T_H = \frac{: J^A J^A :}{k + g_G} - \frac{: J^\alpha J^\alpha :}{k + g_H}, \tag{5.7}$$

where we normal order the current bilinears according to the prescription in [25], we compute that

$$J_\alpha(z) T_{G/H}(w) = \frac{\Delta_{\alpha\beta} J_\beta(w)}{(z-w)^2} + \text{regular}, \tag{5.8}$$

where the dimension matrix  $\Delta_{\alpha\beta}$  is precisely that in (5.6). Hence, we may think that the parafermions  $\Psi_\alpha$  essentially inherit the dimension of the coset generators  $J_\alpha$  with the respect to the energy-momentum tensor  $T_{G/H}$ .

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<sup>6</sup>The analogous construction for the coset  $SL(2, \mathbb{R})/\mathbb{R}$  led to the non-compact parafermions [22].



In our case the group is  $G = \text{SO}(d)$  and the subgroup  $H = \text{SO}(d - 1)$ . According to our normalizations and using the representation matrices for  $\text{SO}(d)$ ,

$$(t_{AB})_{CD} = \sqrt{2}\delta_{C[A}\delta_{B]D}, \tag{5.9}$$

we compute the structure constants as

$$\text{SO}(d) : \quad f_{AB,CD,KL} = \frac{1}{\sqrt{2}}\delta_{BC}\delta_{AK}\delta_{DL} + \text{antisymmetric}, \tag{5.10}$$

from which the dual Coxeter number is  $g_{\text{SO}(d)} = d - 2$ . We split the indices as  $A = (0, i)$ , with  $i = 1, 2, \dots, d - 1$ , being subgroup  $\text{SO}(d - 1)$  indices. Then the algebra of the parafermions (5.5) becomes

$$\{\Psi^i(x), \Psi^j(y)\} = -\frac{k}{\pi}\delta_{ij}\delta'(x - y) - \frac{\pi}{4k}\epsilon(x - y)\left[\delta_{ij}\Psi(x) \cdot \Psi(y) - \Psi_j(x)\Psi_i(y)\right]. \tag{5.11}$$

The absence of linear terms in  $\Psi^i$  on the right-hand side is due to the simple fact that  $\text{SO}(d)/\text{SO}(d - 1)$  is a symmetric space. Thus, structure constants involving only coset space indices are zero. In addition, we have that the bilinear in the structure constants that appears in the dimension formula (5.6) is  $\frac{1}{2}f_{mn,0k,0i}f_{mn,0k,0j} = \frac{1}{2}(d - 2)\delta_{ij}$ , where the factor of  $\frac{1}{2}$  is to avoid overcounting. Hence, the dimension metric for the case of the  $\text{CS}_{d-1,k}$  coset theory is diagonal, i.e.  $\Delta_{ij} = \Delta_k^{\text{CS}_{d-1}}\delta_{ij}$  with

$$\Delta_k^{\text{CS}_{d-1}} = \bar{\Delta}_k^{\text{CS}_{d-1}} = 1 - \frac{d - 2}{2(k + d - 3)}. \tag{5.12}$$

Taking this into account we see that the anomalous dimension of the parafermions exactly cancels that of the vertex operator (5.4). Hence the perturbation is indeed marginal. Note that for  $d = 3$ , corresponding to the  $\text{CS}_2$  model, we get the correct dimension of the quantum parafermions of [21].

Finally, we note that, using the general expressions of [3, 23], the classical parafermions of the  $\text{CS}_{d-1}$  coset theory have the form

$$\Psi^i = \frac{ik}{\pi}\psi^j h_+^{ji}, \quad \bar{\Psi}^i = \frac{ik}{\pi}\bar{\psi}^j h_-^{ji}, \tag{5.13}$$

with

$$h_{\pm}^{-1} = \text{Pe}^{-\int \sigma^{\pm} A_{\pm}}, \tag{5.14}$$

where P stands for path ordering and note that  $h_{\pm}^{-1} = h_{\pm}^T$ . The  $\psi^i$  and  $\bar{\psi}^j$ 's are local expressions of the space variables and first order in the world-sheet derivatives  $\partial_+$  and  $\partial_-$ , respectively and precisely those that appear in the correction term. The path ordered exponentials give to them the non-local character in the space variables. With these definitions the  $\Psi^i$  and  $\bar{\Psi}^i$  are chiral and antichiral, respectively. The  $A_{\pm}$ 's are nothing but the  $\text{SO}(d - 2)$  gauge fields, taking their on-shell values assumed when they are integrated out in the gauged WZW action. Their specific expressions are not relevant to this paper.

With the above general considerations we claim that the perturbation to the action corresponding to the metric (5.1) takes the form

$$\delta\mathcal{L} \sim k V_{d,k}\psi^i\bar{\psi}^i = k V_{d,k}\Psi^i\bar{\Psi}^i, \tag{5.15}$$

where the non-trivial Wilson factors drop out since, on-shell,  $h_+ = h_-$ . In the point particle limit the  $\psi^i$  and  $\bar{\psi}^i$ 's become the differentials  $\xi^i$  and  $S_{ij}\xi^j$ , respectively, thus reproducing the metric perturbation in (5.1).

This kind of marginal perturbation involving dressed parafermions appeared first, to the best of our knowledge, in [17]. In there, the background corresponding to NS5-branes distributed on an ellipsis was explicitly constructed. The small deformation around the uniform distribution of the NS5-branes on a circle, described by an orbifold of the  $SL(2, \mathbb{R})/\mathbb{R} \times SU(2)/U(1)$  direct product conformal cosets [26], is precisely a marginal deformation involving the, appropriately dressed, parafermions of [21]. The existence of a continuous family of exact solutions (circular brane distribution continuously deformed into an elliptical distribution) demonstrates the exactness of the marginal deformation in that case.

In the rest of this section we verify the above general statements for the three- and four-dimensional cases in which explicit results are available. Before proceeding we should stress that the above considerations prove rigorously (i.e. at any finite value of  $k$ ) and generally (i.e. for any  $d$ ) that  $\delta\mathcal{L}$  in (5.15) is a marginal operator of  $CS_{d-1} \times \mathbb{R}_Q$ . However (i) neither these considerations prove that  $\delta ds^2$  in (5.1) always originate from (5.15) — this is the purpose of next subsections for  $d = 3, 4$ ; (ii) nor do they demonstrate that the marginal operator (5.15) is integrable (i.e. exact). A proof of this statement, based on genuine CFT techniques, is beyond the scope of the present paper. Nevertheless, we mention that, when integrated, the marginal deformation of  $CS_{d-1} \times \mathbb{R}_Q$  triggered by the marginal operator at hand is expected to generate only  $CH_d$ , in the lines of thought described in the beginning of section 5 (i.e. by solving the beta-function equations with appropriate boundary conditions), rather than a genuine one-parameter family of conformal models interpolating between  $CS_{d-1} \times \mathbb{R}_Q$  and  $CH_d$ . Indeed, the would-be continuous parameter will be re-absorbed in the redefinition of the radial non-compact coordinate and for any non-zero parameter, the theory will directly match with  $CH_d$ , up to a constant dilaton shift (appearing as an integration constant in the resolution of the beta-function equations.). This phenomenon might look puzzling at a first glance, but many examples exist where it occurs. For instance, in the theory of a non-compact free boson  $\varphi$ , adding the marginal perturbation  $\varepsilon\partial\varphi\bar{\partial}\varphi$  does not affect any physical property because such a modification is eliminated with a field redefinition. A similar though richer situation is met in symmetric and asymmetric parabolic deformations of  $SL(2, \mathbb{R})$ . There the marginal perturbations create an apparent line of deformations that, after re-absorption of the continuous parameter, shrinks onto a new albeit single theory, where only the dilaton (if any) is affected by a parameter-dependent constant shift. The latter shares all background-field asymptotics with the unperturbed theory (see e.g. [7, 9, 27]). We will not further elaborate on this issue here.

## 5.2 The $d = 3$ example and Abelian parafermions

Let's consider the three-dimensional metric (3.1) in the large- $b$  limit, keeping however the

first correction in the  $1/b$  expansion. We find that

$$ds^2 = 2kdx^2 + ds_{(2)}^2 - 4ke^{-2x} [\cos 2\phi (d\theta^2 - \tan^2 \theta d\phi^2) - 4 \tan \theta \sin \phi \cos \phi d\theta d\phi] . \quad (5.16)$$

The extra term with respect to the  $CS_{2,k} \times \mathbb{R}_{Q_{3,k}}$  metric must be a marginal perturbation, as the full three-dimensional model is conformal at every order in perturbation theory in powers of  $1/b$ . Our aim is to rewrite the  $\sigma$ -model action corresponding to this extra term in the metric (5.16) in terms of natural objects in the  $CS_{2,k} \times \mathbb{R}_{Q_{3,k}}$  CFT.

For the  $CS_{2,k}$  factor the natural objects are the classical parafermions [3]. The semi-classical expressions for the chiral parafermions in terms of space variables are (we ignore a factor involving  $k$ )

$$\Psi_{\pm} = (\partial_+ \theta \mp i \tan \theta \partial_+ \phi) e^{\mp i(\phi + \phi_1)} , \quad (5.17)$$

where the phase is

$$\phi_1 = -\frac{1}{2} \int^{\sigma^+} J_+^1 d\sigma^+ + \frac{1}{2} \int^{\sigma^-} J_-^1 d\sigma^- , \quad J_{\pm}^1 = \tan^2 \theta \partial_{\pm} \phi . \quad (5.18)$$

The phase obeys on-shell the condition  $\partial_+ \partial_- \phi_1 = \partial_- \partial_+ \phi_1$  and is well defined, due to the classical equations of motion. Similarly, the expressions for the antichiral ones are

$$\bar{\Psi}_{\pm} = (\partial_- \theta \pm i \tan \theta \partial_- \phi) e^{\pm i(\phi - \phi_1)} . \quad (5.19)$$

It is easy to show that the correction third term in (5.16) can be reproduced by adding to the sigma-model action based on the unperturbed background  $CS_{2,k} \times \mathbb{R}_{Q_{3,k}}$  the term

$$\delta \mathcal{L} = -2k V_{3,k} (\Psi_+ \bar{\Psi}_- + \Psi_- \bar{\Psi}_+) , \quad (5.20)$$

as a perturbation. This takes the form (5.15) and is a  $(1, 1)$  marginal perturbation as we have explicitly shown in the general case.

### 5.3 The $d = 4$ example and non-Abelian parafermions

It turns out to be more convenient to trade the coordinates  $(\hat{b}, \hat{u}, \hat{v})$  introduced in section 3.1 for the angular coordinates  $(\theta, \phi, \omega)$  defined as

$$\hat{b} = \cos 2\theta , \quad \hat{u} = -2 \sin^2 \phi \sin^2 \omega , \quad \hat{v} = 2 \cos^2 \phi . \quad (5.21)$$

This transformation completely covers the range of variables corresponding to the  $CS_3$  compact coset in (3.3). The background metric is

$$ds_{(3)}^2 = d\theta^2 + \tan^2 \theta (d\omega + \tan \omega \cot \phi d\phi)^2 + \frac{\cot^2 \theta}{\cos^2 \omega} d\phi^2 , \quad (5.22)$$

whereas the dilaton reads

$$e^{-2\Phi_{(3)}} = e^{-2\Phi_0} \sin^2 2\theta \sin^2 \phi \cos^2 \omega . \quad (5.23)$$

The chiral parafermions were explicitly computed in [28], a work on universal aspects of string theories, precisely in terms of the variables in the metric (5.22). To conveniently present them for our purposes, let's introduce the forms

$$\begin{aligned}\xi^1 &= \frac{\cot \theta}{\cos \omega} d\phi, \\ \xi^2 &= \cos \omega d\theta - \tan \theta \cot \phi \frac{\sin^2 \omega}{\cos \omega} d\phi - \tan \theta \sin \omega d\omega, \\ \xi^3 &= \sin \omega d\theta + \tan \theta \cot \phi \sin \omega d\phi + \tan \theta \cos \omega d\omega\end{aligned}\tag{5.24}$$

and define the  $\xi_{\pm}^i$ 's via the expansion

$$\xi^i = \xi_+^i d\sigma^+ + \xi_-^i d\sigma^-, \quad i = 1, 2, 3 .\tag{5.25}$$

Then in this basis, the chiral parafermions take the form

$$\psi^i = \xi_+^i, \quad i = 1, 2, 3 .\tag{5.26}$$

As a check, one may verify that  $\Psi^i \Psi^i = \psi^i \psi^i = \xi_+^i \xi_+^i$  is indeed proportional to the component  $T_{++}$  of the energy-momentum tensor corresponding to a sigma-model with metric (5.22). For antichiral ones, we have the slightly different form

$$\bar{\psi}^i = S_{ij} \xi_-^j ,\tag{5.27}$$

where  $S$  is an orthogonal matrix, not connected to the identity, given by

$$S = \begin{pmatrix} -\cos 2\phi & \sin 2\phi & 0 \\ \sin 2\phi & \cos 2\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} .\tag{5.28}$$

Again we note that  $\bar{\Psi}^i \bar{\Psi}^i = \bar{\psi}^i \bar{\psi}^i = \xi_-^i \xi_-^i$  is indeed proportional to the component  $T_{--}$  of the energy-momentum tensor corresponding to a sigma-model with metric (5.22).

Let us now focus on the  $CH_4$  coset. In the asymptotic region where  $b = e^{2x}$  is large, the background fields read (see eqs. (4.1) and (4.2)):

$$ds^2 = 2k dx^2 + ds_{(3)}^2 + \delta ds^2 ,\tag{5.29}$$

where

$$\delta ds^2 = ke^{-2x} \left[ \frac{du^2}{4(u-v)(u-w)} + \left( \frac{(w-v)dv^2}{4(1-v^2)(u-v)} + \frac{(v-w)dw^2}{4(1-w^2)(u-w)} \right) \right] .\tag{5.30}$$

With the help of the coordinate transformation (4.12) together with (5.21), we can recast the subleading term (5.30) as

$$\delta \mathcal{L} = 4k V_{4,k} \psi^i \bar{\psi}^i = 4k V_{4,k} \Psi^i \bar{\Psi}^i ,\tag{5.31}$$

again of the form (5.15). This is a (1, 1) marginal perturbation as we have explicitly shown in the general case.

## 6. Discussion

The main outcome of this work is the appearance of exact  $d$ -dimensional backgrounds  $\mathcal{B}$ , target spaces of gauged WZW models, whose  $(d-1)$ -dimensional “boundaries”  $\partial\mathcal{B}$  are also exact CFT’s. Supplemented with an extra free field with background charge, the theory on  $\partial\mathcal{B}$  admits a marginal operator extracted from the close-to-boundary behaviour of the theory on  $\mathcal{B}$ . This operator is expected to be exact and to reconstruct the theory on  $\mathcal{B}$ . Although we have rigorously proven these statements in the three- and four-dimensional cosets only, the general set-up and the way the proofs work leave little doubt on the validity of the claim in any dimension.

It would be interesting to relate the representation theory of the group  $SO(d, 1)$ , appropriate to the non-compact coset  $CH_d$ , to that of the group  $SO(d)$ , appropriate for the compact coset  $CS_{d-1}$ . We expect that representations similar to the principal series representation of  $SL(2, \mathbb{R})$  will have a limit such that they reduce to  $SO(d)$  representations appropriate for the  $CS_{d-1}$  compact coset. In that spirit, an interesting issue worth the investigation is the propagation of fields in the background  $\mathcal{B}$  in relation to asymptotic or initial data in the remote region  $\partial\mathcal{B}$ . This might help in reconsidering in a truly string framework ideas that have been so far explored only in field theory. It should also be possible to investigate the various issues discussed in the present context in conformal coset theories based on other non-Abelian groups. In that respect, we note the explicit results in [29] for the  $SU(2, 1)/U(2)$  and  $SU(2, 1)/SU(2)$  conformal coset theories.

The backgrounds at hand,  $\mathcal{B}$ ,  $\partial\mathcal{B}$ ,  $\dots$  form hierarchies of gauged WZW models on orthogonal groups. The corresponding target spaces can be Euclidean or Minkowskian, where the  $\partial\mathcal{B}$  is either time-like or space-like. A space-like  $\partial\mathcal{B}$  is interpreted as a collection of data in remote time that evolve toward the future. It would be interesting to analyze the potential cosmological applications of the backgrounds at hand, which provide a CFT generalization of the FRW solutions: in the FRW universes, any spatial section is a maximally symmetric solution of Einstein’s equations, whereas in our case only initial data provide a good CFT.

Finally, we would like to stress once more the emergence of parafermions as building blocks of marginal operators, when appropriately dressed. A proof that these are integrable, based on genuine CFT techniques, would require usage of the (non-) Abelian quantum parafermions. Notice also that marginal operators were usually thought of as products of holomorphic and antiholomorphic currents, absent in gauged WZW (by lack of any residual symmetry) models. To our knowledge, parafermion-based marginal operators appeared only recently, besides the present work, in [17], where their effect was to deform the circular shape of the NS5-brane distribution that generates the background.

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